Technical Notes

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Focusing of a Weak Three-Dimensional Shock Wave

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I. Introduction

THERE have been many studies of focusing shock waves. An important contribution to our understanding of this phenomenon has been provided by Whitham[1,2] in his theory of shock dynamics. The focusing of weak shock waves at caustic surfaces has been described by Guiraud,[3] Hayes,[4] and Pechuzal and Kevorkian.[5] For any smooth shock surface the corresponding caustic surface will always be cusped; this cusp is called an arête (see, e.g., Ref. 6). The focusing of weak shock waves at a two-dimensional arête has been described by Cramer and Seebass.[7] This Note will extend these results to the case of a particular three-dimensional shock surface. Because many of the details resemble those of Ref. 7, we will only report the key results here.

The main difference between the two- and three-dimensional problems is that the shock is simultaneously focusing or defocusing toward two caustic surfaces. The case discussed here considers the focusing of a weak, nearly plane shock at the cusp, i.e., edge of regression, in the first caustic surface encountered by the shock. This will then determine the effect of the initial three-dimensionality on the focusing process. The results are valid if the shock is propagating away from the second caustic sheet; that is, the shock is defocusing with respect to the second sheet. For the case considered here, the focusing process is initially three-dimensional but, in the focal region, becomes essentially two-dimensional. Furthermore, a new similitude not noticed in Ref. 7 is described. This eliminates the need for a similarity parameter and directly relates the value of the pressure coefficient to the geometric and dynamic parameters of the problem.

II. Problem Statement and Results

The coordinate system and typical shock shapes are indicated in Fig. 1. The equation of the initial shock surface is taken to be

$$x=f(y,z)$$

that is, we allow arbitrary variations in the z direction. In order for the shock to focus at a cusp in one of the caustic surfaces, it is assumed that the shock is symmetric with respect to the $y=0$ plane, i.e.,

$$f(y,z)=f(-y,z)$$

for all y and z. The coordinate system can always be defined so that $f(0,0)=f_y(0,0)=f_z(0,0)=0$. It can be shown that the principal radii of curvature at $y=0$, $z=0$ are

$$R_1=[f_{yy}(0,0)]^{-1}>0, \qquad R_2=[f_{zz}(0,0)]^{-1}$$

The shock will focus at the arête in the first caustic sheet provided $R_1<R_2$, whenever $R_2>0$, and when

$$K=\kappa-\frac{3}{2}\frac{[LR_1 f_{yyz}(0,0)]^2}{A}>0$$

where

$$\kappa\equiv L^2 R_1 f_{yyyy}(0,0)>0, \qquad A\equiv\frac{R_2-R_1}{2R_2}>0$$

and L is the length scale associated with the shape of the initial shock surface; we will take this to be defined by the implicit relation $f_{yy}(L,0)=0$. The quantity K is the generalization of the parameter κ found in the two-dimensional theory. This allows for asymmetries in the z direction. We also need to require that $|R_1|\neq|R_2|$ and $K\neq 0$. When these conditions are violated, higher-order singularities result in the linear theory and, therefore, higher pressure levels in the nonlinear theory. The first case corresponds to an axisymmetric arête; this has been treated in a separate study.[8] The second case corresponds to the $\kappa=0$ singularity discussed in the two-dimensional theory.

As in the two-dimensional theory, the results obtained will only be valid if

$$\epsilon\ll\delta^2\ll 1$$

where $\delta\equiv L/R_1$ gives a measure of the rate at which the shock focuses. The parameter ϵ is the value of the pressure coefficient at $t=0$, $y=0$, $z=0$, and immediately following the shock, and gives a measure of the initial strength of the shock.

If we now assume that the shock is initially weak ($\epsilon\ll 1$) and nearly planar ($\delta\ll 1$), and that it propagates into a medium which is uniform and at rest, and if we follow the same basic procedure as in Ref. 7, we find that the flow in the vicinity of the arête in the first caustic surface is governed by the following initial value problem

$$2\hat\phi_{\chi\hat t}+(\gamma+1)\hat\phi_\chi\hat\phi_{\chi\chi}+\hat\phi_{\hat y\hat y}+\hat\phi_{\hat z\hat z}=0$$

where

$$\hat\phi\sim-[(-\hat t)^{3/2}/4K\sqrt{A}]G(\sigma,\Gamma) \tag{1}$$

as $\hat t\to-\infty$. Here G is the integral defined by

$$G(\sigma,\Gamma)\equiv\frac{6}{\Pi}\int_{q_\ell}^{q_u}(-q^4-2q^2+\Gamma q+\sigma)^{1/2}\,dq \tag{2}$$

where

$$\sigma\equiv-\tfrac{2}{3}(\chi/\hat t^2)K, \qquad \Gamma=24[\hat y/(-6\hat t)^{3/2}]K^{1/2}$$

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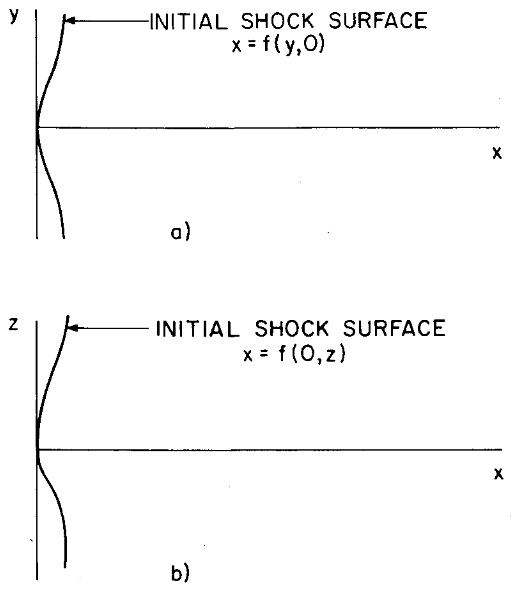

Fig. 1 Coordinate system and two views of a typical shock surface: a) view in $z=0$ plane, shock is symmetric about $y=0$ plane; b) view in $y=0$ plane, shock can be asymmetric in the $z=0$ plane.

and q_u and q_ℓ are the two real roots of

$$-q^4-2q^2+\Gamma q+\sigma=0$$

Thus, $q_u=q_u(\sigma,\Gamma)$, $q_\ell=q_\ell(\sigma,\Gamma)$ only. The quantities $\hat{\phi}$, χ, $\hat{t}$, $\hat{y}$, $\hat{z}$ are related to the dimensional velocity potential ϕ, time t, and position x, y, z by

$$\phi=a_0L\Delta^3\delta^3\hat{\phi}, \qquad x-a_0t=\Delta^2\delta L\chi$$

$$t=(R_1/a_0)(1+\Delta\hat{t}), \qquad y=\Delta^{3/2}L\hat{y}$$

and

$$z=\Delta^{3/2}L\hat{z}$$

where a_0 is the undisturbed sound speed and $\Delta\equiv\epsilon^{2/3}\delta^{-4/3}=o(1)$.

We now note that the initial condition in Eq. (1), i.e., the incoming wave, is independent of z. We may therefore assume that $\hat{\phi}\neq\hat{\phi}(\hat{z})$ and drop the term involving the $\hat{z}$ derivativesin the differential equation in Eq. (1). Thus, even though the flow is initially three-dimensional, it may be considered two-dimensional in the focal region. In fact, the similitude described below shows that this flow is identical to the two-dimensional one once the independent and dependent variables are scaled appropriately.

At this stage we could rescale the variables in Eq. (1) to obtain a similitude similar to that obtained in Ref. 7. We would find a similarity parameter Q which, in the present study, would be given by $[(\gamma+1)/2](KA)^{-1/3}$. However, a new similitude has been found that does not involve a similarity parameter. We rescale the variables in Eq. (1) as follows

$$\chi\equiv(a^2/K)\xi, \qquad \hat{t}\equiv a\tau,$$

$$\hat{y}\equiv(a^{3/2}/K)\eta, \qquad \hat{\phi}\equiv(a^{3/2}/4K\sqrt{A})\Phi$$

where

$$a\equiv\left[\frac{\gamma+1}{2}\,\frac{K}{2\sqrt{A}}\right]^{2/3}$$

Then Eqs. (1) and (2) become

$$2\Phi_{\xi\tau}+\Phi_\xi\Phi_{\xi\xi}+\Phi_{\eta\eta}=0 \tag{3}$$

with

$$\Phi\sim-(-\tau)^{3/2}G(\sigma,\Gamma)$$

as $\tau\to-\infty$, where, in terms of the scaled variables, σ and Γ read

$$\sigma=-\tfrac{2}{3}(\xi/\tau^2) \qquad \text{and} \qquad \Gamma=24\eta/(-6\tau)^{3/2}$$

Thus, $\Phi=\Phi(\xi,\eta,\tau)$ only and the solution to Eq. (3) can describe the flow for any two- or three-dimensional shock wave, at least within the limitations of the present study. The pressure coefficient $c_p\approx2\phi_X/a_0$, where $X\equiv x-a_0t$, can now be written

$$c_p=\frac{\epsilon^{2/3}\delta^{2/3}}{[2(\gamma+1)KA]^{1/3}}\Phi_\xi(\xi,\eta,\tau) \tag{4}$$

Here we shall note two special cases of interest. The first is where the shock is also symmetric about the $z=0$ plane. We then have $f_{yyz}(0,0)=0$ and $K=\kappa$; thus, Eq. (4) becomes

$$c_p=\frac{\epsilon^{2/3}\delta^{2/3}}{[(\gamma+1)\kappa]^{1/3}}\left(\frac{R_2}{R_2-R_1}\right)^{1/3}\Phi_\xi(\xi,\eta,\tau)$$

This is clearly the simplest three-dimensional case we can consider. The main effect of the three-dimensionality is contained in the term involving R_1 and R_2; δ and κ can be calculated without knowing the variation in the shock shape in the z direction. As would be expected, the overall pressure levels tend to be increased if, in addition to focusing toward the arête, the shock is focusing toward the second caustic sheet as well, i.e., $R_2>R_1>0$, and these levels are decreased if it is defocusing relative to the second sheet, i.e., $R_2<0$; here we have a quantitative measure of this effect. The second special case is that of a purely two-dimensional focusing; this is obtained by taking the limit $f_{yyz}(0,0)\to0$ and $R_2/R_1\to0$, resulting in

$$c_p=\frac{\epsilon^{2/3}\delta^{2/3}}{[(\gamma+1)\kappa]^{1/3}}\Phi_\xi(\xi,\eta,\tau)$$

This is seen to be completely consistent with the results of Ref. 7.

Inspection of Eq. (4) indicates the importance of the similitude. The actual pressure distribution and, in particular, the maximum value of the pressure coefficient will be determined either from numerical solutions of Eq. (3) subject to the appropriate jump conditions or by experiment. In either case, the function $\Phi_\xi(\xi,\eta,\tau)$ is determined by a single numerical solution or experiment rather than a series of these for various values of a similarity parameter. It should also be clear that the experiment need only involve a two-dimensional shock surface; the results for a three-dimensional surface can be obtained by a simple rescaling. For example, if the maximum value of the pressure coefficient is determined experimentally for a two-dimensional shock wave having $K=\kappa_e$, $\delta=\delta_e$ and a strength given by ϵ_e, then Eq. (4) may be used to show that the maximum pressure coefficient for a three-dimensional focusing in the same gas is given by

$$c_{pm}=\left(\frac{\epsilon}{\epsilon_e}\right)^{2/3}\left(\frac{\delta^2\kappa_e}{\delta_e^2K2A}\right)^{1/3}c_{pm}\Big|_e$$

where the terms without subscripts correspond to the three-dimensional shock. In an analogous manner, Eq. (4) along with the scalings for ξ, η, and τ may be used to relate the pressure distribution obtained in a two-dimensional experiment or a numerical solution of Eq. (3) to that for a three-dimensional focusing problem.

In conclusion, the focusing of a weak, three-dimensional shock wave at a cusp in a caustic has been studied. It has been shown that although the focusing is initially three-dimensional, the flow in the vicinity of the arête is essentially two-dimensional. The details of the flow are governed by Eq. (3); once these are obtained, the pressure distribution is given by Eq. (4). It has also been shown that the similarity parameter used in Ref. 7 was unnecessary and, as a result, Eq. (4) delineates the dependence of the resultant pressure levels on the initial shape and strength of the shock. For three-dimensional shock waves satisfying the conditions discussed here, the similitude also allows us to determine once and for all the pressure distribution simply by analyzing, either experimentally or numerically, a two-dimensional focusing problem.

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Approach to Self-Preservation in Plane Turbulent Wakes

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Introduction

THE purpose of this Note is to examine the manner in which moderate Reynolds number, plane turbulent wakes behind wake generators of different shapes approach a unique self-preserving state, and to point out what appear to be some surprising features of the process.

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Experiments

Table 1 lists details of the several wake generators used in the experiments discussed here. Except for two cases,[1,5] all of the wakes were generated in an open-circuit suction-type wind tunnel with a contraction ratio of about 10, and a 30 cm square, 4.27 m long test section. The wind speed was constant to within about 1.5% and the freestream turbulence was about 0.15%.[2] All mean velocity measurements were made with a round pitot tube of 1 mm o.d. using a micromanometer capable of reading 0.05 mm alcohol; no corrections for finite turbulence levels were attempted for the pitot measurements.

Background

By self-preservation, we mean here that the mean velocity and the Reynolds shear stress distributions must be independent of the streamwise position when normalized by the same velocity and length scales. In the asymptotic limit of vanishing velocity defect, a two-dimensional self-preserving (linear) turbulent wake is characterized by constant values of two parameters defined in Ref. 7 as $W=(w_0/U)\sqrt{(x/\theta)}$ and $\Delta=\delta/\sqrt{x\theta}$. Here, w_0 is the maximum of the velocity defect w, δ is the half-wake thickness given by the distance from the centerplane to where the defect is half the maximum, x and y are, respectively, the distances from the trailing edge of the wake generator and from the wake centerplane, and θ is the momentum thickness defined by

$$\theta=\int_{-\infty}^{\infty}(w/w_0)(1-w/w_0)\mathrm{d}y \qquad (1)$$

If the asymptotic self-preserving state is unique, the parameters W and Δ must assume universal values, say W^* and Δ^*. As appropriate to small but finite defect wakes, the nature of correction terms to W^* and Δ^* cannot be assessed on the basis of linear theory alone, but it was argued in Ref. 8 that the correction terms are of $0(w_0/U)$. Thus, the behavior of the measured values of W and Δ against w_0/U gives us a gross indication of the manner in which the unique asymptotic state (if one exists) is approached.

Results

Figures 1 and 2 show, respectively, the variation of the parameters W and Δ with w_0/U; corresponding variation in x/θ ranges typically from about 10 to about 1000. (Where necessary, convergence corrections according to the suggestions of Refs. 2 and 9 have been applied to the data.) Although it is not surprising that different wakes approach self-preservation through different routes, the degree of variability and the non-monotonic behavior shown by W and Δ was unexpected. It should be emphasized that both δ/θ and w_0/U showed monotonic variations with x/θ for all of these wakes. Although all wakes seem to approach the asymptotic values W^* and Δ^* indicated on the figures (more about which will be said shortly), thus indicating an approach to *the* self-preservation state, there are substantial differences among them even when the defect ratio is as low as 5%: the large eddies seem to remember the manner of their generation even so far downstream! The wake behind a twin-plate generator appears to have the simplest behavior and attains self-preservation in the shortest distance (as was indeed found by Narasimha and Prabhu,[7] who first used it) probably because large eddies in the flow are rendered weak by the nature of the mean strain field that occurs there.

The relatively simple behavior of the wake parameters in a twin-plate generator wake led us to make detailed far-wake measurements with the sole purpose of determining the asymptotic values W^* and Δ^*. These were determined by extrapolating linearly to zero defect the parameters W and Δ obtained from measurements in the region of small but finite defect. The chief conclusion of these measurements, reported elsewhere,[8] can be stated as $W^*=1.63\pm0.02$, and